J. Fluid Mech. (1991), *vol.* 232, *pp.* 285–301
Printed in Great Britain

An asymptotic theory of near-field propeller acoustics

By N. PEAKE AND D. G. CRIGHTON
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street Cambridge CB3 9EW, UK

(Received 24 November 1990)

This paper presents expressions for the harmonic components of the *near-field* acoustic pressure of a B-bladed unswept single-rotation propeller. These are derived using asymptotic approximations to the standard radiation integrals for steady loading and thickness noise, under the assumption that B is large. The dependence of the pressure on blade operating conditions (both supersonic and subsonic) is described by simple formulae, which provide significant insights into the mechanisms of sound generation by rotating bodies. For supersonic motion, the importance of sources satisfying the Ffowcs Williams & Hawkings sonic condition is demonstrated, whilst for subsonic blades the near-field noise is proved to be tip-dominated. Expressions for the noise (valid from close to the tips right out to infinity) are given in both cases, requiring matching across an Airy function smoothing region when the tips move subsonically. Excellent agreement between the asymptotic formulae and both full numerical evaluations (with a considerable saving in CPU time) and experimental data is achieved.

1. Introduction

Since the realization that certain advanced ultra-high-bypass ratio engines (e.g. 'propfans' and ducted fans) represent a potentially significant fuel saving over the more conventional jet engines currently powering civilian aircraft, there has been intense interest in the aerodynamics and aeroacoustics of high-speed propellers. The most pressing noise issues include the questions of whether such aircraft would meet tight environmental noise certification requirements, and also match current expectations of cabin levels. Therefore, given the vast development and testing costs involved, it is acutely necessary to develop accurate theoretical prediction schemes, and great progress has been made in this direction, both in the frequency domain (Hanson 1980, 1983) and the time domain (Farassat 1981).

Most of the existing analysis is based on an integral formula for the acoustic pressure (due to Ffowcs Williams & Hawkings 1969), which reduces the problem to one of determining the sound field generated by a surface distribution of sources associated with thickness and steady loading effects (calculation of the latter being effected using steady aerodynamic codes), and a volume distribution of nonlinear quadrupoles. Evaluation of the resulting frequency-domain radiation integrals numerically, at least in the near field, and for the higher harmonics of greater subjective importance, is exceedingly expensive in CPU time, owing to the presence of a rapidly oscillating Bessel function representing the radial acoustic efficiency. More importantly, no matter how rapidly such numerical evaluations can be performed, they cannot yield any information on the underlying physics or scaling laws. However, great simplification is possible when the blade number B is assumed large, and has been used by Parry & Crighton (1989), who performed an asymptotic analysis as $B \to \infty$, and developed expressions for the far-field thickness and steady loading noise. Close agreement with more exact (and time-consuming) methods was found for B as small as 4, and even more so for modern values of 7 or 8, and the scheme is now in day-to-day use at Rolls-Royce plc. As well as being useful for prediction, the scheme provides the basis for *control*, because it leads to the identification of the location of the dominant noise sources and can be used to produce simple scaling laws useful in design considerations.

Parry & Crighton's analysis was, however, performed under the assumption that the observer–hub separation, R_0, was large, and it therefore considerably under-estimates near-field levels, so important for considerations of passenger comfort and structural integrity. In this paper, the authors describe how the asymptotic approximation can be employed in the near field, and will in fact derive formulae which are uniformly valid in observer position, at least close to the propeller disk plane, and which therefore include the far-field results as a special case.

In §2 a brief outline is given of the derivation of the steady loading and thickness radiation integrals, in observer-fixed coordinates, and of their manipulation into a form suitable for asymptotic analysis. It should be noted that these formulae rely on the 'thin-blade approximation', as used by Hanson (1980) (i.e. the thickness and steady loading noise sources are moved from the genuine blade surface onto the mid-chord). The limitations of this method are discussed in Peake & Crighton (1990, 1991*b*); essentially, the assumption of vanishingly small blade thickness can be employed in estimating the amplitude of the lower harmonics of blade passing frequency, but is unreliable for predicting the highest harmonics, where the Doppler-contracted wavelength is comparable with the blade thickness. It is therefore envisaged that the formulae described in this paper will be of particular practical use in calculation of the sound pressure level and of tones with moderate harmonic number; the more complicated expressions given in Peake & Crighton (1991*b*) must be used if accurate predictions of the real-time wave form (involving the calculation of a large number of harmonics to resolve the impulsive pressure peaks) are required. The subsequent asymptotic approximations for supersonic and subsonic blades are described in §§3 and 4 respectively; for a supersonic blade, the Ffowcs Williams & Hawkings sonic, or 'Mach', condition is confirmed, whilst in the subsonic case, the field away from the blades is seen to possess an Airy smoothing region (analogous to a caustic in ray theory), which separates regions of near-field and far-field behaviour. A further refinement to the supersonic expansion is described in §5. In this paper the quadrupole terms (assumed small at moderate forward speeds and for thin blades) will be ignored; the reader is referred to Peake & Crighton (1991*a*) for a discussion of asymptotic theory applied to the quadrupoles.

Good agreement is obtained with both numerical methods and experiment, both in terms of trends with varying design parameters, and of absolute level predictions. A detailed comparison of various near-field corrections with test data is presented by Boyd & Peake (1990). The large-blade-number approximation is again seen to provide important physical insight, over and above the full numerical solution, although it should be again stressed that it is a consistent and accurate prediction scheme in its own right.

2. Derivation of radiation integrals

In this section a derivation will be presented of the radiation integral for the steady loading noise of a B-bladed single-rotation propeller. This result has been previously stated by a number of authors, notably Garrick & Watkins (1954), and in a different form by Hanson (1983), but the procedure used here parallels most closely the work of Parry & Crighton (1986), whose far-field formulation will be generalized.

The starting point is an expression for the acoustic pressure p at observer position (x, y, z) in a medium at rest with density ρ_0 and sound speed c_0, due to a point force $\boldsymbol{F}$, whose position in reception coordinates is (x_1, y_1, z_1),

$$p = -\frac{1}{4\pi}\nabla\cdot\left[\frac{\boldsymbol{F}(t-\sigma/c_0)}{S}\right], \tag{1}$$

where
$$S = \{(x-x_1)^2+\beta^2[(y-y_1)^2+(z-z_1)^2]\}^{\frac{1}{2}}, \tag{2}$$

$$\sigma = \frac{M_x(x-x_1)+S}{\beta^2} \tag{3}$$

is the retarded-time observer–source separation, $\beta = (1-M_x^2)^{\frac{1}{2}}$ and M_x is the axial (flight) Mach number.

The reception polar coordinates (R_0, Θ_0) are made clear in figure 1(*a*), and the blade cross-section in figure 1(*b*). The observer and blade-element coordinates, relative to the hub, are

$$\left.\begin{aligned}(x, y, z) &= (R_0\cos\Theta_0, R_0\sin\Theta_0, 0),\\ (x_1, y_1, z_1) &= (-cX\cos\alpha, r\cos(\phi+\psi),\ r\sin(\phi+\psi)),\end{aligned}\right\} \tag{4}$$

where ϕ is the azimuthal angle and the blade section has been taken as vanishingly thin. Following Parry & Crighton (1986), the force due to *all* the B blade elements at radial station r and chordwise station X is expressed in the form

$$\mathrm{d}\boldsymbol{F}\left(X, r, \phi, t-\frac{\sigma}{c_0}\right) = \frac{B}{4\pi}\rho_0 c c_L U_\mathrm{r}^2 F(X)\sum_{m=-\infty}^{\infty}\exp\left[\mathrm{i}mB\left(\phi+\Omega t-\frac{\Omega\sigma}{c_0}\right)\right]$$
$$\times\ (\sin\alpha, -\cos\alpha\sin(\phi+\psi), \cos\alpha\cos(\phi+\psi))\,\mathrm{d}X\,\mathrm{d}r, \tag{5}$$

with U_r the helical velocity, $F(X)$ a normalized shape function defining the chordwise blade loading, and where ψ is related to the chordwise non-compactness and angle of twist α by

$$\psi = \frac{cX}{r}\sin\alpha. \tag{6}$$

The local section lift coefficient is c_L, the local chord c, and X runs from $-\frac{1}{2}$ at the leading edge of a section to $+\frac{1}{2}$ at the trailing edge. Substituting (5) into (1), integrating over all blade elements and making the simple transformation $\phi\rightarrow\phi-\psi$ yields an integral expression for P_m, the mth harmonic of the steady loading noise,

$$P_m \sim \frac{\mathrm{i}\Omega mB^2c_0\rho_0}{16\pi^2}\int_{r_\mathrm{h}}^{D/2}\int_0^{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}}\exp(\mathrm{i}mB[\phi+\Omega t-M_\mathrm{t}\tilde{\sigma}-\psi])$$
$$\times\left\{\frac{M_x\sin\alpha}{\beta^2}+\frac{(x-x_1)\sin\alpha}{S\beta^2}-\frac{(y-y_1)\cos\alpha\sin\phi}{S}+\frac{(z-z_1)\cos\alpha\cos\phi}{S}\right\}\frac{M_\mathrm{r}^2cc_L}{S}F(X)\,\mathrm{d}X\,\mathrm{d}\phi\,\mathrm{d}r, \tag{7}$$

where a tilde denotes normalization with respect to blade length, D is the propeller diameter, and r_h is the radius of the hub and M_t and M_r are respectively the tip

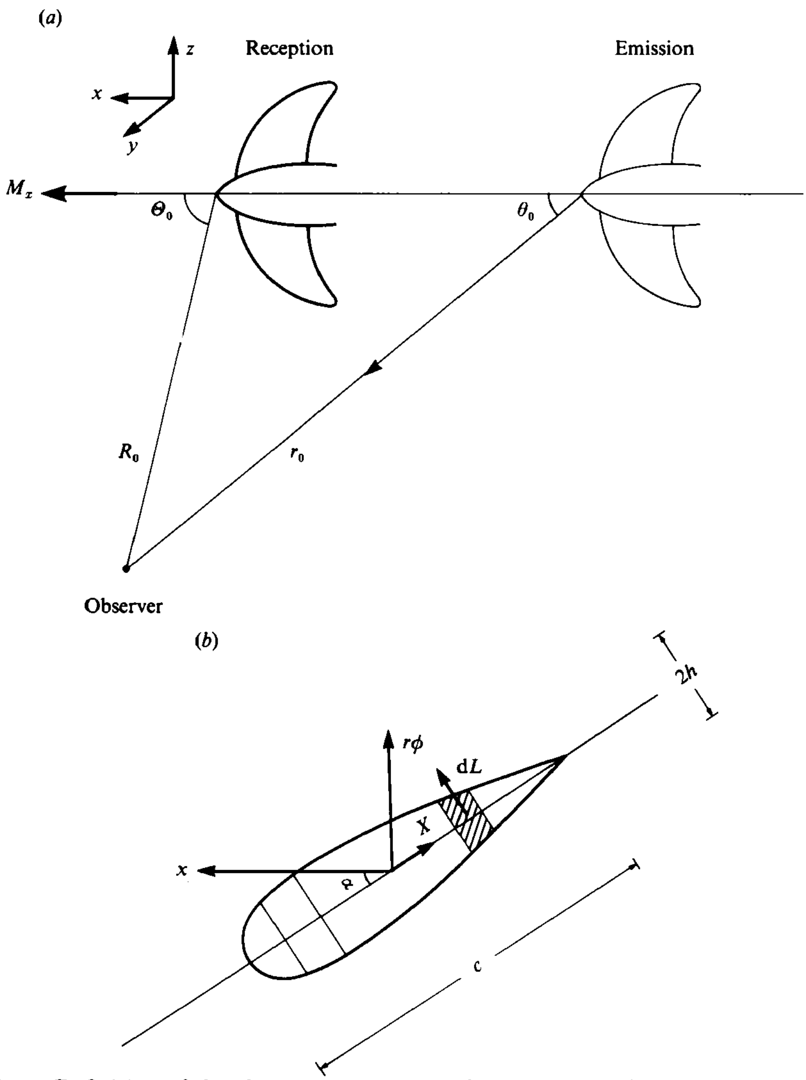


FIGURE 1. (a) Definition of the observer reception and emission coordinates, and the axial and tip Mach numbers. The observer is positioned in the horizontal (x, y)-plane. (b) The blade cross-section.

rotational and helical Mach numbers. In deriving (7) a term in S^{-3}, arising from differentiation of S^{-1} in (1), has been neglected as being $O(mB)^{-1}$ smaller than the others, and therefore insignificant at distances from the tip of practical interest (at least for the relatively large modern values of B). It is again emphasized that, as a result of our thin-blade approximation, (7) will only be valid for moderate values of m (and will therefore be perfectly adequate for calculating the sound pressure level of a subsonic propeller); for the higher harmonics required to evaluate the fine details of the time-domain wave form, radiation integrals taking full account of blade thickness are needed, derivation of which is described by Peake & Crighton (1990, 1991*b*).

In order to suppress the effect of chordwise non-compactness, one sets $c = 0$ in the exponential of (7), equivalent to taking the leading term in a small-c expansion (valid only for the moderate values of harmonic number m employed in this paper), and since F integrates to unity, the radiation integral reduces to a double integral over the propeller disk plane (now with normalized radial station z, and z_0 the value of z at the hub),

$$P_m \sim \frac{\mathrm{i}\Omega mB^2 c_0 \rho_0}{16\pi^2} \int_{z_0}^{1} \int_{0}^{2\pi} M_{\mathrm{r}}^2 c c_L \exp\left(\mathrm{i}mB[\phi+\Omega t - M_{\mathrm{t}}\tilde{\sigma}]\right) \times \frac{1}{\tilde{S}} \left\{ \frac{M_x \sin\alpha}{\beta^2} + \frac{\tilde{R}_0 \cos\Theta_0 \sin\alpha}{\tilde{S}\beta^2} - \frac{\tilde{R}_0 \sin\Theta_0 \cos\alpha \sin\phi}{\tilde{S}} \right\} \mathrm{d}\phi \, \mathrm{d}z, \tag{8}$$

which is Garrick & Watkins' (1954) result for the sound due to a regular B-element fan of concentrated line sources. Crighton & Parry (1991 *a*) describe a method for including finite chord length in the asymptotic scheme, but for simplicity the concentration distribution will be adopted in the rest of this paper. Equation (8) can be manipulated into a form amenable to asymptotic analysis by noting that the second term is easily related to the derivative of the first with respect to $\tilde{R}_0 \cos\Theta_0$ (denoted $\tilde{x}$), at least to leading order in mB, and the ϕ-integral in the third term can be integrated by parts, to give a term very similar to the first, again to leading order in mB. Equation (8) then reduces to

$$P_m \sim \frac{\mathrm{i}\Omega mB^2 c_0 \rho_0}{16\pi^2} \left\{ \frac{\mathrm{i}\beta^2}{mBM_{\mathrm{t}} M_x} \frac{\partial}{\partial \tilde{x}} I^{(1)} + I^{(2)} \right\}, \tag{9}$$

with $I^{(1)}$ and $I^{(2)}$ defined by

$$\left.\begin{aligned} I^{(1)} &= \int_{z_0}^{1} \int_{0}^{2\pi} \frac{M_x M_{\mathrm{r}}^2 \sin\alpha}{\tilde{S}\beta^2} c c_L \exp\left(\mathrm{i}mB[\phi+\Omega t - M_{\mathrm{t}}\tilde{\sigma}]\right) \mathrm{d}\phi \, \mathrm{d}z, \\ I^{(2)} &= -\int_{z_0}^{1} \int_{0}^{2\pi} \frac{M_{\mathrm{r}}^2 \cos\alpha}{\tilde{S} M_{\mathrm{t}} z} c c_L \exp\left(\mathrm{i}mB[\phi+\Omega t - M_{\mathrm{t}}\tilde{\sigma}]\right) \mathrm{d}\phi \, \mathrm{d}z. \end{aligned}\right\} \tag{10}$$

The asymptotic expansions of $I^{(1)}$ and $I^{(2)}$ are very similar, so the problem has essentially been reduced to that of approximating just one integral. It should be noted that only the $m > 0$ components need be calculated; those for $m < 0$ can be found simply by complex conjugation. For definiteness, it will be assumed in the rest of this paper that $\alpha = \tan^{-1}(zM_{\mathrm{t}}/M_x)$. In practice, of course, each airfoil will be aligned at a small (non-zero) angle of attack, thereby generating lift. However, since, as will be demonstrated later, the noise generation will be dominated by a single radial station, the error involved in using our choice of α will be small; in any event, any given twist distribution can easily be included in our final asymptotic formulae.

An expression for the thickness noise can be obtained in a similar manner, starting from the pressure due to a point mass source $m(t)$ (see Lighthill 1962)

$$p = \frac{m(t-\sigma/c_0)}{4\pi S}, \tag{11}$$

which yields the radiation integral, again for chordwise compact sources,

$$\left.\begin{aligned} P_m &\sim \frac{m^2 B^3 \Omega^2 \rho_0}{8\pi^2} \left(1 + \frac{\mathrm{i}M_x}{mBM_{\mathrm{t}}} \frac{\partial}{\partial \tilde{x}}\right)^2 I^{(3)}, \\ I^{(3)} &= \int_{z_0}^{1} \int_{0}^{2\pi} \frac{hc}{\tilde{S}} \exp\left(\mathrm{i}mB[\phi+\Omega t - M_{\mathrm{t}}\tilde{\sigma}]\right) \mathrm{d}\phi \, \mathrm{d}z, \end{aligned}\right\} \tag{12}$$

where the blade is assumed symmetric, with maximum local thickness $2h(r)$. In what follows, attention will be restricted to the steady loading noise, although of course the thickness noise analysis proceeds in exactly the same way.

3. Asymptotic analysis: a supersonic propeller

An asymptotic expansion of the radiation integrals of §2 will now be sought in terms of suitably rapidly decreasing functions of B, in the limit $B \rightarrow \infty$. For simplicity, consider the integral

$$\left. \begin{aligned} I &= \int_{z_0}^{1} \int_{0}^{2\pi} \frac{\exp(\mathrm{i}mB[f(\phi, z)])}{\tilde{S}} \, \mathrm{d}\phi \, \mathrm{d}z, \\ f(\phi, z) &= \phi - \frac{M_\mathrm{t} \tilde{S}}{\beta^2}, \end{aligned} \right\} \tag{13}$$

where $$\tilde{S} = \{\tilde{R}_0^2 \cos^2 \Theta_0 + \beta^2 [\tilde{R}_0^2 \sin^2 \Theta_0 + z^2 - 2z\tilde{R}_0 \sin \Theta_0 \cos \phi]\}^{\frac{1}{2}}, \tag{14}$$

which is closely related to $I^{(1)}$, $I^{(2)}$ and $I^{(3)}$. To make an asymptotic approximation we must seek stationary points of the argument of the exponential, i.e. points (z_*, ϕ_*) such that

$$\nabla f = 0. \tag{15}$$

The analysis falls into two quite distinct cases, depending on whether stationary points exist within the range of integration or not. When they are present, standard stationary phase theory predicts that I will be dominated by contributions from the particular azimuthal positions (z_*, ϕ_*), in which case the noise is supersonic in character (cf. Crighton & Parry 1991*a*). If no stationary point exists, or equivalently the roots z_* are all greater than unity, it will be shown that the noise is tip-dominated, and essentially subsonic in character. The two cases will be considered separately; in this section the leading-order term in the asymptotic expansion is only calculated when a *single* stationary point is present, and in §5 a second-order modification is given. The subsonic case will be treated in §4.

Suppose first, then, that I does possess stationary phase points within its range of integration, and so (15) reduces to the two equations

$$z_* = \tilde{R}_0 \sin \Theta_0 \cos \phi_* \tag{16}$$

and $$1 = \frac{M_\mathrm{t} \tilde{R}_0 \sin \Theta_0 \, z_* \sin \phi_*}{\tilde{S}}, \tag{17}$$

which are most readily interpreted by transformation to emission coordinates, via

$$\tilde{S} = \tilde{\sigma}(1 - M_x \cos \theta), \quad \sin \theta = \frac{\tilde{R}_0 \sin \Theta_0 \sin \phi}{\tilde{\sigma}}, \tag{18}$$

where θ is the angle between the source–observer vector and the direction of forward flight, at emission time, and is therefore a function of z and ϕ. One can then easily form the identity

$$M_\mathrm{t} z_* \sin \theta + M_x \cos \theta = 1, \tag{19}$$

which means that the velocity component of the dominant sources in the direction of the observer is exactly sonic, in agreement with the Ffowcs Williams & Hawkings (1969) theory. This is shown in figure 2.

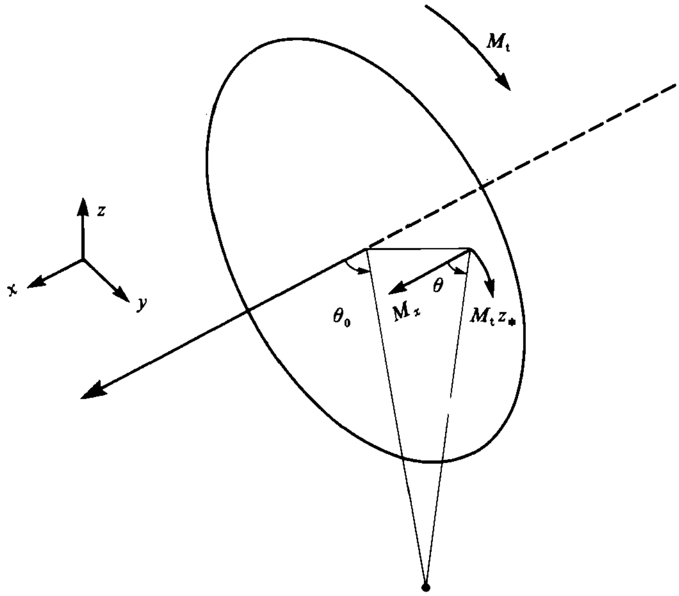


FIGURE 2. The dominant source of noise for a supersonic propeller, located at the Mach radius, and with an exactly sonic velocity component in the observer direction.

Equations (16) and (17) have been solved exactly, to give

$$z_*^2 = \frac{\pm(\beta^4 + M_t^4 \tilde{R}_0^4 \sin^4 \Theta_0 - 2\beta^2 M_t^2 \tilde{R}_0^2 \sin^2 \Theta_0 - 4M_t^2 \tilde{R}_0^2 \cos^2 \Theta_0)^{\frac{1}{2}} + \beta^2 + M_t^2 \tilde{R}_0^2 \sin^2 \Theta_0}{2M_t^2}, \tag{20}$$

and ϕ_* is then recovered from (16). These roots are only admissible as stationary points of equation (13) provided that both the conditions $|\cos \phi_*| < 1$ and $z_0 < z_* < 1$ are satisfied, and for different observer positions there will be two, one or no stationary points within the range of integration. An explicit expression for the leading term in the asymptotic expansion of I then follows from standard stationary phase analysis of two-dimensional integrals (see Jones 1982), and involves calculation of the Hessian matrix of f, given by the formula

$$\left.\begin{aligned} \det \boldsymbol{H} &= \frac{M_t^2 z_*^2}{\tilde{S}_*^2}\left(\frac{\tilde{R}_0^2 \cos^2 \Theta_0}{\tilde{S}_*^2} - \tan^2 \phi_*\right), \\ \tilde{S}_*^2 &= \tilde{R}_0^2 \cos^2 \Theta_0 + \beta^2 \tilde{R}_0^2 \sin^2 \Theta_0 \sin^2 \phi_*. \end{aligned}\right\} \tag{21}$$

For simplicity, it will now be assumed that $\cos \Theta_0$ is small, corresponding to an observer positioned close to the propeller plane (in the region of highest acoustic pressure), and that the observer is outboard of the blade tips. In this case it turns out that just one of the roots in (20) (the smaller one) is admissible, and the circle $z = z_*$ is termed the *Mach radius*; note that z_* is a function of Θ_0. Then, to leading order in $\cos \Theta_0$, the stationary point becomes

$$\left.\begin{aligned} \cos \phi_* &= \frac{\beta}{M_t \tilde{R}_0} + O(\cos^2 \Theta_0), \\ z_* &= \frac{\beta}{M_t} + O(\cos^2 \Theta_0). \end{aligned}\right\} \tag{22}$$

In other words, for an observer positioned close to the plane of the propeller, the noise will be dominated by the Mach radius, provided that $\beta/M_t < 1$, which will be taken to be the condition that the *noise is supersonic in character* (at least for an observer close to the disk plane). Proceeding with the stationary phase calculation then yields, correct to $O(\cos\Theta_0)$, and for $m > 0$,

$$I \sim \frac{2\pi}{mB\beta[(M_t^2\tilde{R}_0^2/\beta)-1]^{\frac{1}{2}}}\exp\left\{imB\left[\phi_* - \left(\frac{M_t^2\tilde{R}_0^2}{\beta^2}-1\right)^{\frac{1}{2}}\right]\right\}. \tag{23}$$

Substitution back into (9), with the observation that $I^{(1)} \sim -I^{(2)}$, gives a final value for the steady loading noise, to leading order in B^{-1} and $\cos\Theta_0$, of

$$P_m^{(1)} \sim \frac{iB\Omega c_0\rho_0}{8\pi}\frac{\cos\Theta_0 cc_L}{\beta^3}\left(\frac{M_t^2\tilde{R}_0^2}{\beta^2}-1\right)^{-\frac{1}{2}}\left(1-\frac{\beta^2}{M_t^2\tilde{R}_0^2}\right)^{-\frac{1}{2}}$$
$$\times\exp\left(imB\left[\Omega t - \frac{M_t M_x\tilde{R}_0\cos\Theta_0}{\beta^2}+\phi_* - \left(\frac{M_t^2\tilde{R}_0^2}{\beta^2}-1\right)^{\frac{1}{2}}\right]\right), \tag{24}$$

where the quantities c and c_L are evaluated at the Mach radius and the superfix (1) denotes the first term in the asymptotic expansion. This expression retains full dependence on all the important design parameters, and is clearly very much easier to compute than the original integrals in (10) (this is true even for arbitrary observer angle, a more complicated formula for which would follow in exactly the same way).

Equation (24) was derived on the assumption that $\tilde{R}_0$ is of order unity (this was implicit in our stationary phase analysis). However, if the limit $\tilde{R}_0 \to \infty$ is now taken, Parry & Crighton's (1989) far-field result is regained, and it is therefore seen that (24) is uniformly valid for any observer–hub separation, containing the far field as a special case. This should be of great advantage in prediction work.

Finally, it should be noted that the Fourier series defined by (24) does not converge, and only exists in the generalized sense. The corresponding time-domain result can be regained by use of Lighthill's (1958) theory of Fourier transforms, from which it can be seen that the pressure field contains a series of delta functions, which is exactly what one would expect, given that the chordwise loading distribution has here been treated as a delta function. A method to smooth out this singularity for more diffuse loading has been described by Crighton & Parry (1991 *a*).

4. Asymptotic analysis: a subsonic propeller

When the propeller is rotating subsonically, so that no point along its length can have a sonic velocity component in the observer direction (or equivalently z_*, the root of (16) and (17), is greater than unity) a different approach must be adopted.

The identity (Erdelyi *et al.* 1954), representing the free-space Green's function for the Helmholtz equation as a Fourier integral,

$$\frac{\exp[-ib(a^2+y^2)^{\frac{1}{2}}]}{(a^2+y^2)^{\frac{1}{2}}} = -\frac{i}{2}\int_{-\infty}^{\infty} H_0^{(2)}[a(b^2-x^2)^{\frac{1}{2}}]\exp(ixy)\,dx, \tag{25}$$

with $-\pi < \arg(b^2-x^2)^{\frac{1}{2}} \leqslant 0$; $a > 0$, will be used together with the Bessel function addition theorem (Gradshteyn & Ryzhik 1980)

$$H_0^{(2)}[c(z^2-2z\tilde{R}_0\sin\Theta_0\cos\phi+\tilde{R}_0^2\sin^2\Theta_0)^{\frac{1}{2}}] = \sum_{n=-\infty}^{\infty} J_n(cz)\,H_n^{(2)}(c\tilde{R}_0\sin\Theta_0)\exp(in\phi), \tag{26}$$

assuming that $\tilde{R}_0 \sin \Theta_0 > z$ (so that only the sound directly ahead of the propeller is thereby excluded from consideration).

Then I can be rewritten in the form

$$I = -\frac{\mathrm{i}mBM_t\pi}{\beta^2}\int_{z_0}^{1}\int_{-\infty}^{\infty} \mathrm{J}_{mB}\left[mB\frac{zM_t}{\beta}(1-k^2)^{\frac{1}{2}}\right] \times \mathrm{H}^{(2)}_{mB}\left[mB\frac{M_t\tilde{R}_0\sin\Theta_0}{\beta}(1-k^2)^{\frac{1}{2}}\right]\exp\left(\mathrm{i}mBk\tilde{R}_0\cos\Theta_0\frac{M_t}{\beta^2}\right)\mathrm{d}k\,\mathrm{d}z, \quad (27)$$

where k is a helical wavenumber along the propeller advance path. One would now proceed by splitting up the z and k integration ranges into separate regions, such that a single asymptotic expression for the special functions pertains in each region, and then evaluating each term separately. This approach could of course have been used in §3, but it was felt more natural to use the stationary phase analysis where possible, especially since the stationary point has such an obvious physical interpretation.

In order to fix ideas, it is helpful to consider the special case $\cos\Theta_0 = 0$. Under this simplifying assumption it can be shown that the dominant contribution to I comes from the neighbourhood of $z = 1$ (by integration of the z-integral once by parts) and from the neighbourhood of $k = 0$ (owing to the presence of a saddle point at the k-origin). When the assumption of $\Theta_0 = \frac{1}{2}\pi$ is dropped, one can show that the z-integral is still dominated by the endpoint $z = 1$, but that the saddle point has been displaced off the real line, and the k-contour must be deformed accordingly. Calculation of this saddle point can in general only be done numerically so, as previously, the calculation will be restricted to determination of the leading-order term in $\cos\Theta_0$.

Thus, following the procedure outlined above, it can be shown, after some effort, that the dominant contribution to I comes from the term

$$I \sim -\frac{\mathrm{i}mBM_t\pi}{\beta^2}\int_{z_0}^{1}\int_{-1+\delta}^{1-\delta} \mathrm{J}_{mB}\left[mB\frac{zM_t}{\beta}(1-k^2)^{\frac{1}{2}}\right] \times \mathrm{H}^{(2)}_{mB}\left[mB\frac{M_t\tilde{R}_0}{\beta}(1-k^2)^{\frac{1}{2}}\right]\exp\left(\mathrm{i}mBk\tilde{R}_0\cos\Theta_0\frac{M_t}{\beta^2}\right)\mathrm{d}k\,\mathrm{d}z, \quad (28)$$

where δ is some small positive parameter, and the next term in the expansion is $O(mB)^{-1}$ smaller than the first.

The evaluation of (28) is accomplished by use of the large-argument–large-order expansion (Abramowitz & Stegun 1968)

$$\mathrm{J}_{mB}(mB\operatorname{sech}\gamma) \sim \frac{\exp\left(mB(\tanh\gamma-\gamma)\right)}{(2\pi mB\tanh\gamma)^{\frac{1}{2}}}, \quad (29)$$

where γ is defined by

$$\operatorname{sech}\gamma = \frac{M_t z}{\beta}(1-k^2)^{\frac{1}{2}}, \quad (30)$$

and of course $M_t/\beta < 1$ for a subsonic propeller.

The z-integral can then be integrated once by parts (so that the leading-order term comes from the neighbourhood of $z = 1$), demonstrating that subsonic propeller noise is, throughout the near field and the far field, tip-dominated, to give

$$I \sim -\frac{\mathrm{i}\pi M_t}{\beta^2}\int_{-1+\delta}^{1-\delta}\frac{1}{\tanh\gamma_t(2\pi mB\tanh\gamma_t)^{\frac{1}{2}}}\mathrm{H}^{(2)}_{mB}\left[mB\frac{M_t\tilde{R}_0}{\beta}(1-k^2)^{\frac{1}{2}}\right] \times\exp\left(mB(\tanh\gamma_t-\gamma_t)\right)\exp\left(\mathrm{i}mBk\tilde{R}_0\cos\Theta_0\,M_t/\beta^2\right)\mathrm{d}k, \quad (31)$$

where γ_t is just γ evaluated at the tip. Terms in $\cos^2\Theta_0$ have been neglected.

The subsequent evaluation of I falls into three main categories, depending on the value of $M_t \tilde{R}_0/\beta$, as follows:

(a) $\tilde{R}_0$ is strictly greater than β/M_t;

(b) $\tilde{R}_0$ is strictly less than β/M_t;

(c) $\tilde{R}_0$ lies within a band of width $(mB)^{-\frac{2}{3}}$ around β/M_t.

Region (c) is a smoothing region between region (a) (which extends outwards from the Mach radius to infinity) and (b) (which extends inwards to close to the tips); it will be shown that in fact (c) exhibits an Airy function dependence analogous to that found across a focus in ray theory. In each of the above regions the Hankel function in (31) has a different asymptotic form, and these match together smoothly as the observer makes a radial traverse, but must be considered separately, as follows.

(a) When the observer is outside the Mach radius, the asymptotic approximation

$$\mathrm{H}^{(2)}_{mB}(mB\sec\alpha) \sim \left(\frac{2}{\pi mB}\right)^{\frac{1}{2}} \frac{\exp(\mathrm{i}mB[\alpha-\tan\alpha]+\frac{1}{4}\mathrm{i}\pi)}{(\tan\alpha)^{\frac{1}{2}}}, \tag{32}$$

where
$$\sec\alpha = \frac{M_t\tilde{R}_0}{\beta}(1-k^2)^{\frac{1}{2}}, \tag{33}$$

is substituted into (31), and it is then seen that the saddle point satisfies $F_1'(k_s) = 0$, where

$$F_1(k) = \tanh\gamma_t - \gamma_t - \mathrm{i}\tan\alpha + \frac{\mathrm{i}M_t\tilde{R}_0\cos\Theta_0}{\beta^2}. \tag{34}$$

This could be solved numerically for any Θ_0, but a strictly algebraic solution only proves possible to a given order in $\cos\Theta_0$, so that

$$k_s = \frac{\mathrm{i}\tilde{R}_0\cos\Theta_0}{M_t(\tilde{R}_0^2-1)}(\tanh\gamma_0+\mathrm{i}\tan\alpha_0) + O(\cos^2\Theta_0), \tag{35}$$

the suffix 0 denoting evaluation at $k=0$. Standard saddle point theory then provides the leading term of I (here for $m>0$), namely

$$I \sim \left(\frac{2\pi}{(mB)^3}\right)^{\frac{1}{2}} \frac{M_t}{\beta^2} \frac{1}{(\tanh\gamma_0)^{\frac{3}{2}}(\tan\alpha_0)^{\frac{1}{2}}(\tan\alpha_0+\mathrm{i}\tanh\gamma_0)^{\frac{1}{2}}} \times \exp\left(mB\left[\tanh\gamma_0-\gamma_0+\mathrm{i}\alpha_0-\mathrm{i}\tan\alpha_0+\frac{\mathrm{i}M_t k_s\tilde{R}_0\cos\Theta_0}{\beta^2}\right]\right). \tag{36}$$

Derivation of the pressure field is completed by substitution back into (9), noting that

$$\left.\frac{\partial F}{\partial\tilde{x}}\right|_{k=k_s} = \frac{\mathrm{i}M_t}{\beta^2}k_s \tag{37}$$

and
$$I^{(2)} \sim -\frac{\beta^2}{M_t^2}I^{(1)}, \tag{38}$$

to yield

$$P_m \sim \frac{\mathrm{i}\Omega B(mB)^{-\frac{1}{2}}c_0\rho_0}{2^{\frac{7}{2}}\pi^{\frac{3}{2}}\beta^4}\left\{M_x\left(1-\frac{\beta^2}{M_t^2}\right)-k_s\right\}\frac{M_t^2M_r cc_L}{(\tanh\gamma_0)^{\frac{3}{2}}(\tan\alpha_0)^{\frac{1}{2}}(\tan\alpha_0+\mathrm{i}\tanh\gamma_0)^{\frac{1}{2}}} \times \exp\left(mB\left[\mathrm{i}\Omega t+\tanh\gamma_0-\gamma_0+\mathrm{i}\alpha_0-\mathrm{i}\tan\alpha_0-\mathrm{i}M_tM_x\frac{\tilde{R}_0\cos\Theta_0}{\beta^2}\right]\right). \tag{39}$$

The $m < 0$ components are obtained from their positive counterparts by complex conjugation. Again, this expression represents a great simplification over the radiation integral (even if one had to calculate k_s numerically), but still retains full dependence on the important design parameters. The harmonics decay exponentially rapidly, consistent with the absence of a stationary point, and, just as in the supersonic case, the formula is uniformly valid, from just outside the Mach radius, right out to infinity.

It should be noted that (39) has an $x^{-\frac{1}{4}}$ singularity as the observer approaches the Mach radius. However, the authors stress that this is not a real singularity of the pressure field, and is smoothed out by consideration of an observer position sufficiently close to the Mach radius, as in subsection (*c*).

Finally, it has been assumed that the loading is non-zero towards the tip, so that the c_L in (39) is simply evaluated at $z = 1$. A simple modification to include the more realistic effect of (typically parabolic) decay in loading has been described by Parry & Crighton (1989). Essentially, if it is supposed that the lift coefficient takes the form $c_L(1-z)^{\nu}$ as $z \to 1$, then an additional factor

$$\frac{\nu!}{(mB \tanh \gamma_0)^{\nu}} \tag{40}$$

is introduced in (39) (and similarly in (45), (50), (51) and (52)).

(*b*) When the observer is inside the Mach radius, one must use the different approximation

$$\mathrm{H}^{(2)}_{mB}\,(mB \operatorname{sech} \alpha) \sim \frac{\mathrm{i} \exp\,(mB(\alpha - \tanh \alpha))}{(\frac{1}{2}\pi mB \tanh \alpha)^{\frac{1}{2}}}, \tag{41}$$

where
$$\operatorname{sech} \alpha = \frac{M_t \tilde{R}_0}{\beta}(1-k^2)^{\frac{1}{2}}. \tag{42}$$

The saddle point again satisfies an equation of the form $F_2'(k_s) = 0$, where

$$F_2(k) = \alpha - \tanh \alpha + \tanh \gamma_t - \gamma_t + \frac{\mathrm{i} M_t k \tilde{R}_0 \cos \Theta_0}{\beta^2}, \tag{43}$$

and to leading order is

$$k_s = -\frac{\mathrm{i} \tilde{R}_0 \cos \Theta_0}{M_t(\tilde{R}_0^2 - 1)}(\tanh \gamma_0 + \tanh \alpha_0) + O(\cos^2 \Theta_0). \tag{44}$$

Hence, using (31), (38) and (9), an explicit expression for the leading term of the pressure can be found to be

$$P_m \sim \frac{\mathrm{i}\Omega B(mB)^{-\frac{1}{2}} c_0 \rho_0}{\beta^4 2^{\frac{7}{2}} \pi^{\frac{3}{2}}} \left\{ M_x \left(1 - \frac{\beta^2}{M_t^2}\right) - k_s \right\} \frac{M_t^2 M_r c c_L}{(\tanh \gamma_0)^{\frac{3}{2}} (\tanh \alpha_0)^{\frac{1}{2}} (\tanh \gamma_0 - \tanh \alpha_0)^{\frac{1}{2}}}$$
$$\times \exp\left(mB \left[\mathrm{i}\Omega t + \tanh \gamma_0 - \gamma_0 + \alpha_0 - \tanh \alpha_0 - \mathrm{i} M_t M_x \frac{\tilde{R}_0 \cos \Theta_0}{\beta^2} \right] \right). \tag{45}$$

Again, this expression is singular like $x^{-\frac{1}{4}}$ as $M_t \tilde{R}_0/\beta$ approaches unity from below, and is smoothed out by the theory described in subsection (*c*). As one would expect, p becomes infinite as the tip, $\tilde{R}_0 = 1$, is approached.

(c) Consideration must now be given as to how to match the outer and inner solutions found in (a) and (b) respectively. Suppose that the observer lies within $O(mB)^{-\frac{2}{3}}$ of the Mach radius, and write

$$\frac{M_t \tilde{R}_0}{\beta} = 1 + y(mB)^{-\frac{2}{3}}, \tag{46}$$

where y is an $O(1)$ parameter; then the appropriate Hankel function expansion is

$$\mathrm{H}^{(2)}_{mB}(mB(1+y(mB)^{-\frac{2}{3}})) \sim \frac{2^{\frac{1}{3}}}{(mB)^{\frac{1}{3}}}[\mathrm{Ai}\,(-2^{\frac{1}{3}}y) + \mathrm{i}\,\mathrm{Bi}\,(-2^{\frac{1}{3}}y)]. \tag{47}$$

In other words, $\mathrm{H}^{(2)}_{mB}$ contains no exponential dependence on mB in this region, so that the saddle point of the exponential in (31) now satisfies $F_3'(k_s) = 0$, where F_3 is simply defined by

$$F_3(k) = \tanh\gamma_t - \gamma_t + \frac{\mathrm{i}M_t k}{\beta^2}\tilde{R}_0 \cos\Theta_0. \tag{48}$$

This is solved to leading order as

$$k_s = \frac{\mathrm{i}M_t \tilde{R}_0 \cos\Theta_0}{\beta^2 \tanh\gamma_0} + O(\cos^2\Theta_0), \tag{49}$$

and thus the pressure in the vicinity of the Airy smoothing region is found to be

$$P_m \sim -\frac{\Omega c_0 \rho_0 B(mB)^{-\frac{1}{3}} M_r c c_L}{2^{\frac{11}{3}}\pi\beta^2}\left\{M_x + \frac{k_s}{\tanh^2\gamma_0}\right\}[\mathrm{Ai}(-2^{\frac{1}{3}}y) + \mathrm{i}\,\mathrm{Bi}\,(-2^{\frac{1}{3}}y)]$$
$$\times \exp\,(mB[\mathrm{i}\Omega t + \tanh\gamma_0 - \gamma_0 - \mathrm{i}M_t M_x \tilde{R}_0 \cos\Theta_0/\beta^2]). \tag{50}$$

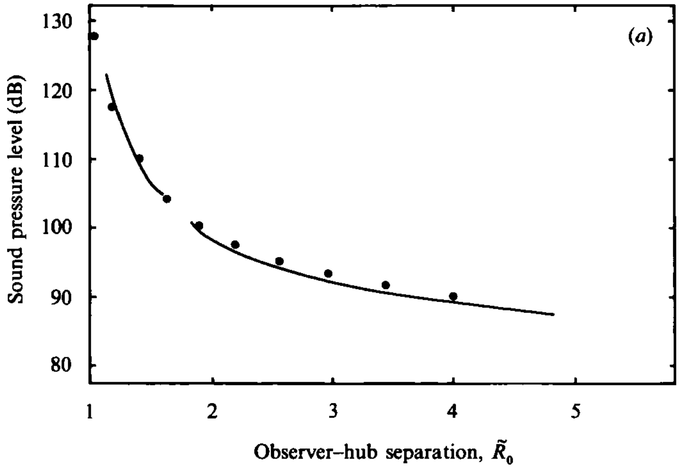
This is the solution required to provide a matching between (a) and (b) in the sense of matched asymptotic expansion theory. The first term in the outer expansion of (50) (in this case with $\cos\Theta_0 = 0$ for simplicity) is found to be

$$-\frac{\mathrm{i}\Omega c_0 \rho_0 M_x M_r c c_L B(mB)^{-\frac{1}{2}} \exp\,(-\frac{1}{4}\mathrm{i}\pi)}{\beta^2 \pi^{\frac{3}{2}} 2^{\frac{15}{4}}} \frac{1}{[(M_t \tilde{R}_0/\beta) - 1]^{\frac{1}{4}}}$$
$$\times \exp\left(mB\left[\mathrm{i}\Omega t - \frac{2^{\frac{3}{2}}}{3}\mathrm{i}\left(\frac{M_t \tilde{R}_0}{\beta} - 1\right)^{\frac{3}{2}} + \tanh\gamma_0 - \gamma_0\right]\right) \tag{51}$$

when $\tilde{R}_0 > \beta/M_t$, and

$$-\frac{\mathrm{i}\Omega c_0 \rho_0 M_x M_r c c_L B(mB)^{-\frac{1}{2}}}{\beta^2 \pi^{\frac{3}{2}} 2^{\frac{15}{4}}} \frac{1}{[1 - (M_t \tilde{R}_0/\beta)]^{\frac{1}{4}}}$$
$$\times \exp\left(mB\left[\mathrm{i}\Omega t + \frac{2^{\frac{3}{2}}}{3}\left(1 - \frac{M_t \tilde{R}_0}{\beta}\right)^{\frac{3}{2}} + \tanh\gamma_0 - \gamma_0\right]\right) \tag{52}$$

when $\tilde{R}_0 < \beta/M_t$. It can easily be verified that these are equal to the first terms in the inner expansions of (39) and (45) respectively. Composite expansions can now be formed, which retain the accuracy of the individual expansions, but which provide a single expression valid in both inner and outer regions. Inside the Mach radius the composite is formed by adding (45) and (50), and subtracting off their common part (52), whilst outside the Mach radius the composite expansion is found from adding (39) and (50) and subtracting (51).



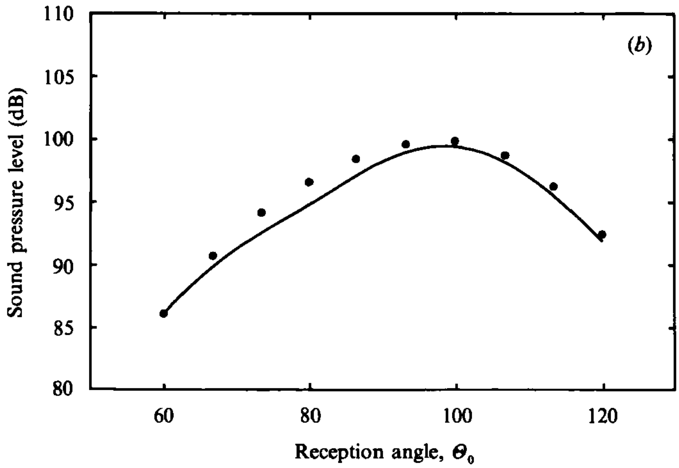


FIGURE 3. (a) A comparison between the (outer) asymptotic solution away from the Mach radius (solid line) and a full numerical integral (●), for the first harmonic of the loading noise generated by a 7-bladed propeller. A radial traverse is made in the plane of the propeller ($\Theta_0 = \frac{1}{2}\pi$), with $M_t = 0.57, M_x = 0.2$ and $\tilde{c} = 0.08$. The inner solution has not been plotted, accounting for the gap in the curve across the narrow Airy function transition region. (b) A comparison between the asymptotic solution outside the Mach radius (solid line) and a full numerical integration (●), for the first harmonic of the loading noise generated by a 7-bladed propeller. The reception angle Θ_0 is varied, at a constant sideline distance of $\tilde{R}_0 \sin \Theta_0 = 2$, with other conditions as in (a).

An expression has therefore been constructed for the acoustic pressure generated by a subsonic propeller, valid for observer angle Θ_0 close to $\frac{1}{2}\pi$ and from close to the tips right out to infinity. Outside the Mach radius, the sound pressure level is essentially far field in character, but inside near-field effects become strong, and a $1/r$-type decay (as in the far-field analysis of Parry & Crighton 1989) would substantially underpredict the pressure, particularly under take-off conditions. Very good agreement with exact numerical evaluation of equation (8) has been achieved (even for the moderate value of $B = 7$), shown for a radial traverse in figure 3(a), and for a fieldshape in figure 3(b) (plotting fieldshapes in fact requires inclusion of $O(\cos^2 \Theta_0)$ terms in (39) or (45), but these can easily be found); the latter demonstrates that our small-$\cos \Theta_0$ approximation is valid over a useful range. For convenience, the lift

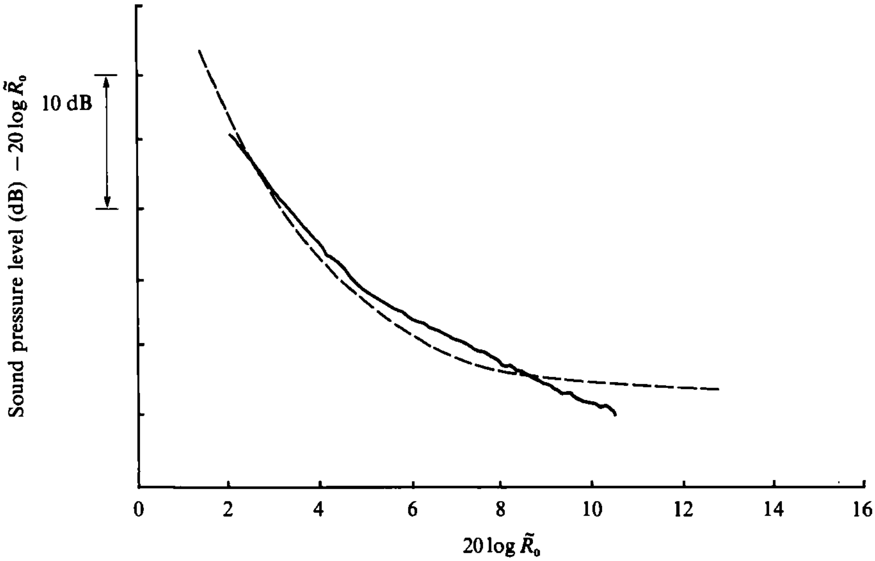


FIGURE 4. A comparison between wind tunnel test data (solid curve) and the (outer) asymptotic solution (broken curve) for a radial traverse in the plane of a 7-bladed model propfan. The relevant parameters (typical of take-off) include $M_x = 0.2$ and $M_t = 0.57$; the quantity sound pressure level minus $20\log\tilde{R}_0$ has been plotted against $20\log\tilde{R}_0$, so that in the far field the plots asymptote to horizontal straight lines. In this case the outer solution has been interpolated across the Mach radius using a simple curve fitting routine.

coefficient is supposed here to take the constant value $c_L = 0.25$ along almost all the blade span, and to decay to zero parabolically between $z = 0.95$ and $z = 1$. Of course, the noise due to any other form of tip loading could be found equally easily, for suitable choice of parameters in (40). However, the values chosen for M_t, M_x, $\tilde{c}$ and B correspond to a realistic (take-off) design condition. It is again emphasized that the full numerical solution in the near field is very expensive in CPU time, whereas the asymptotic calculation is virtually trivial. Moreover, determination of higher harmonics numerically would prove even more difficult, whereas the asymptotic method can be used to generate harmonics of arbitrarily high order with ease (provided of course that suitable allowance is made for non-zero blade thickness, as in Peake & Crighton 1990), and avoiding all the problems associated with numerical stability and convergence. In figure 4 a comparison is made between wind tunnel test data (obtained on a $\frac{1}{5}$th-scale model propfan rig; see Kirker 1990 for full experimental details) and the asymptotic solution. A radial traverse in the plane of the propeller is shown (now with a simple curve-fitting routine applied across the narrow inner region); it should be emphasized that, since the value of the lift coefficient at the tip was not available to us, a one-parameter fit between the prediction and the data is made, but one this has been done, excellent agreement between the measured and predicted variations with radial separation is achieved. Further comparisons with experimental data are described by Boyd & Peake (1990).

The composite solution is plotted in figure 5; the rather large value of $B = 75$ (accounting for the negligible sound pressure level) is required to produce a (virtually) smooth plot, although this value could no doubt be reduced if higher terms in the various asymptotic expansions were calculated. For *practical* prediction purposes, however, use of just the outer expansion (as in figure $3a$), with only a small loss of information in the vicinity of the Mach radius, is quite adequate.

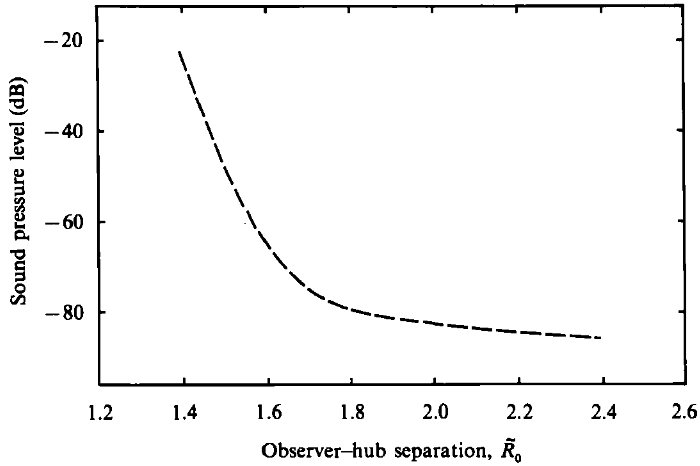


FIGURE 5. The composite asymptotic solution for $B = 75$, conditions as in figure 3(*a*).

5. Higher approximation for the supersonic propeller

In their work on the far-field noise of a supersonic propeller, Crighton & Parry (1991*a*, *b*) proved that the leading-order noise contribution comes from the point on the blades at the Mach radius, confirmed for the near field in §2 of this paper, and that the second term, $O(mB)^{-\frac{1}{2}}$ lower than the first, represents a tip effect. Their calculation will now be repeated in the near field, in order to improve the accuracy of predictions made using (24). The easiest way to do this is by consideration of the Bessel function expansion of I in (27), rather than attempting to generalize the stationary phase approach. It should be emphasized that for the subsonic blade, knowledge of the first term is quite sufficient, since the second is $O(mB)^{-1}$ smaller.

For simplicity, consider only the case $\cos\Theta_0 = 0$ here, although the analysis could be generalized in much the same way as before, at least to leading order. Using (27), and following Crighton & Parry (1991*b*), we split the z-integral as

$$\int_{z_0}^{z^-} + \int_{z^-}^{z^+} + \int_{z^+}^{1}, \tag{53}$$

where

$$z^{\pm} = \frac{\beta}{M_t}\left(1 \pm \frac{(mB)^{-\frac{1}{3}}}{2}\right), \tag{54}$$

so that the Mach radius is contained in a band of width $O(mB)^{-\frac{1}{3}}$. It can then be shown that the leading-order term in I comes from the neighbourhood of $k = 0$, and from the second term in (53), and is equivalent to the stationary phase analysis of §3. Further contributions from the endpoints of the first and third z-integrals above arise, but those from $z = z^+$ and $z = z^-$ cancel (cf. Crighton & Parry 1991*b*), so that the second-order contribution to P_m comes from the tip. This is calculated by use of the asymptotic result

$$\mathrm{J}_{mB}(mB\sec\gamma) \sim \left(\frac{1}{2\pi mB\tan\gamma}\right)^{\frac{1}{2}}\{\exp\left(\mathrm{i}mB\left[\tan\gamma-\gamma\right]-\tfrac{1}{4}\mathrm{i}\pi\right)+\text{complex conjugate}\} \tag{55}$$

with

$$\sec\gamma = \frac{zM_t}{\beta}(1-k^2)^{\frac{1}{2}}, \tag{56}$$

in addition to (32). The analysis follows in very much the same way as in §4; an integration by parts yields the tip contribution, provided the loading does not vanish there, and this is followed by use of the saddle point method at $k=0$, to yield a final expression for the second-order term $p^{(2)}$ ($O(mB)^{-\frac{1}{2}}$ smaller than the first-order term),

$$P_m^{(2)} \sim \frac{\mathrm{i}\Omega B(mB)^{-\frac{1}{2}}c_0\rho_0 M_x M_t^2 M_r c c_L}{8\pi^2\beta^4}\frac{\exp(\mathrm{i}mB[\Omega t+\alpha_0-\tan\alpha_0-\frac{1}{4}\mathrm{i}\pi])}{(\tan\gamma_0)^{\frac{3}{2}}(\tan\alpha_0)^{\frac{1}{2}}} \times\left\{\frac{\exp(\mathrm{i}mB[\tan\gamma_0-\gamma_0])}{(\tan\alpha_0-\tan\gamma_0)^{\frac{1}{2}}}-\frac{\exp(\mathrm{i}mB[\gamma_0-\tan\gamma_0])}{(\tan\alpha_0+\tan\gamma_0)^{\frac{1}{2}}}\right\} \tag{57}$$

(with c and c_L evaluated at the tip), which can then be added onto $P_m^{(1)}$ in (24). The modification for vanishing tip loading is easily calculated.

6. Conclusions

This paper has described how asymptotic theory based on the idea of large blade number can be used in calculation of the near-field noise of a propeller, and represents a major simplification over full numerical solutions, without any significant loss in accuracy. Closed algebraic formulae for the pressure field have been derived, providing both important insights into the underlying physics and invaluable scaling laws for design purposes, and in good agreement with both full numerical evaluations and experimental data. Work is now well under way towards the completion of an entire prediction scheme for propeller and fan noise, relying on this kind of asymptotic analysis, and including all the major source mechanisms present in ultra-high-bypass ratio engines.

The authors are grateful to C. J. Chapman for a number of helpful discussions. The data used in plotting figure 4 are the property of Rolls-Royce plc. N. P. acknowledges receipt of an SERC studentship and the generous support of Rolls-Royce plc.

REFERENCES

Abramowitz, M. & Stegun, I. A. 1968 *Handbook of Mathematical Functions*. Dover.

Boyd, W. K. & Peake, N. 1990 An approximate method for the prediction of propeller near-field effects. *AIAA Paper* 90–3998.

Crighton, D. G. & Parry, A. B. 1991*a* Asymptotic theory of propeller noise – Part II: Supersonic single-rotation propeller. *AIAA J.* (to appear).

Crighton, D. G. & Parry, A. B. 1991*b* Higher approximations in the asymptotic theory of propeller noise. *AIAA J.* (to appear).

Erdelyi, A., Magnus, W., Oberhettinger, F. & Tricomi, F. G. 1954 *Tables of Integral Transforms*. McGraw-Hill.

Farassat, F. 1981 Linear acoustic formulae for calculation of rotating blade noise. *AIAA J.* **19**, 1122–1130.

Ffowcs Williams, J. E. & Hawkings, D. L. 1969 Sound generated by turbulence and surfaces in arbitrary motion. *Phil. Trans. R. Soc. Lond.* A **264**, 321–342.

Garrick, I. E. & Watkins, C. E. 1954 A theoretical study of the effect of forward speed on the free-space sound-pressure field around propellers. *NACA Rep.* 1198.

Gradshteyn, I. S. & Ryzhik, I. M. 1980 *Tables of Integrals, Series and Products*. Academic.

Hanson, D. B. 1980 Helicoidal surface theory for harmonic noise of propellers in the far field *AIAA J.* **18**, 1213–1220.

Hanson, D. B. 1983 Compressible helicoidal surface theory of propeller aerodynamics and noise. *AIAA J.* **21**, 881–889.

JONES, D. S. 1982 *The Theory of Generalised Functions.* Cambridge University Press.

KIRKER, T. J. 1990 Procurement and testing of a 1/5 scale advanced counter rotating propfan model. *AIAA Paper* 90–3975.

LIGHTHILL, M. J. 1958 *An Introduction to Fourier Analysis and Generalised Functions.* Cambridge University Press.

LIGHTHILL, M. J. 1962 The Bakerian Lecture. Sound generated aerodynamically. *Proc. R. Soc. Lond.* A **267**, 147–182.

PARRY, A. B. & CRIGHTON, D. G. 1986 Theoretical prediction of single-rotation propeller noise. *AIAA Paper* 86–1891.

PARRY, A. B. & CRIGHTON, D. G. 1989 Asymptotic theory of propeller noise – Part I: Subsonic single-rotation propeller. *AIAA J.* **27**, 1184–1190.

PEAKE, N. & CRIGHTON, D. G. 1990 Radiation integrals for sound generation by the Lighthill quadrupoles in propeller acoustics. *AIAA Paper* 90–3993.

PEAKE, N. & CRIGHTON, D. G. 1991*a* Lighthill quadrupole radiation in supersonic propeller acoustics. *J. Fluid Mech.* **223**, 363–382.

PEAKE, N. & CRIGHTON, D. G. 1991*b* Helicoidal coordinate integrals for radiation from high-speed propellers. *J. Fluid Mech.* To be submitted.